THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2058 Honours Mathematical Analysis I 2022-23
Homework 2 solutions
27th September 2022

- Homework will be posted on both the course webpage and blackboard every Tuesday. Students are required to upload their solutions on blackboard by 23:59 p.m. next Tuesday. Additional announcement will be made if there are no homework that week.
- Please send an email to echlam@math.cuhk.edu.hk if you have any questions.

1. (b) For any $\epsilon > 0$, consider
$$\left|\frac{2n}{n+2} - 2\right| = \frac{4}{n+2}.$$
If we pick by Archimedean property $N_0 \in \mathbb{N}$ so that $N_0 > \frac{4}{\epsilon}$, then for any $n \geq N_0$, we have
$$\left|\frac{2n}{n+2} - 2\right| < \frac{4}{n} \leq \frac{4}{N_0} < \epsilon.$$
(d) For any $\epsilon > 0$, consider
$$\left|\frac{(-1)^n n}{n^2+1}\right| = \frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n}.$$
We may pick by Archimedean property an $N_0 \in \mathbb{N}$ so that $N_0 > \frac{1}{\epsilon}$, then for any $n \geq N_0$, we have
$$\left|\frac{(-1)^n n}{n^2+1}\right| < \frac{1}{n} \leq \frac{1}{N_0} < \epsilon.$$

2. Let (x_n) and (y_n) be convergent sequences, we will show that $u_n = \max\{x_n, y_n\}$ also defines a convergent sequence. Write $x = \lim x_n$ and $y = \lim y_n$, we will prove that $\lim u_n = \max\{x, y\}$. We will prove this separately for the cases when $x = y$ and $x \neq y$.

   If $x = y = L$, then for any $\epsilon > 0$, we may find $N_1, N_2 \in \mathbb{N}$ so that $|x_n - L| < \epsilon$ for $n \geq N_1$ and $|y_n - L| < \epsilon$ for $n \geq N_2$. Now for $n \geq \max\{N_1, N_2\}$, we have
$$|u_n - L| = |\max\{x_n, y_n\} - L| \leq \max\{|x_n - L|, |y_n - L|\} < \epsilon.$$
   If $x \neq y$, without loss of generality we may assume $x > y$, then given $\epsilon > 0$, replace it with a smaller ϵ' if necessary, we may assume that $x - 2\epsilon > y$ still. Then by convergence of (x_n) and (y_n), we have $N_1, N_2 \in \mathbb{N}$ so that $|x_n - x| < \epsilon$ for $n \geq N_1$ and $|y_n - y| < \epsilon$ for $n \geq N_2$. Then for $n \geq N = \max\{N_1, N_2\}$, we have
$$x_n - y_n = (x_n - x) + (y - y_n) + (x - y) > -|x_n - x| - |y_n - y| + 2\epsilon > -\epsilon - \epsilon + 2\epsilon = 0.$$
   In other words, for n big enough, we always have $x_n > y_n$. And hence $u_n = \max\{x_n, y_n\} = x_n$ for $n \geq N$. Clearly $\lim u_n = x$ exists in this case.

   For the sequence $v_n = \min\{x_n, y_n\}$, simply consider $v_n = -\max\{-x_n, -y_n\}$ and apply the previous part to see that limit exists.

3. We may prove that x_n is convergent by showing that it is monotonic increasing and has an upper bound.

   First, note that x_n is always positive because it is the square root of a positive number. Suppose that $2 > x_n$, we will prove that it implies $2 > x_{n+1}$. This is simply because $x_{n+1} = \sqrt{x_n + 2} < \sqrt{2+2} = 2$. Inductively, since $x_1 = 1$, we see that $2 > x_n$ for all $n \in \mathbb{N}$. Also note that x_n is always positive, in particular this implies $x_{n+1} = \sqrt{2 + x_n} > \sqrt{2} \geq 1$ for all $n > 1$.

   Now $x_{n+1} > x_n$ if and only if $x_{n+1}^2 > x_n^2$. Consider $x_{n+1}^2 - x_n^2 = x_n + 2 - x_n^2 = (2 - x_n)(x_n + 1)$ is greater than 0 if and only if $-1 < x_n < 2$, which holds by the argument above. Therefore the sequence is monotone increasing.

   By monotone convergence theorem, (x_n) is convergent with limit equals to $\sup\{x_n : n \in \mathbb{N}\}$. In particular, if $L = \lim x_n$, it must satisfies $L = \sqrt{L+2}$ by the recurrence relation. Therefore $L^2 - L - 2 = 0$, i.e. $L = 2$ or $L = -1$. The negative root cannot be the limit since all $x_n > 0$. Therefore $L = 2$.